THE FIRST FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY FOR A RECTANGULAR PARALLELEPIPED

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In 1852 Lamé [1] formulated the first fundamental problem of the theory of elasticity for a rectangular parallelepiped. An approximate solution to this problem was given by Filonenko-Borodich [2 and 3] who used Castigliano's variational principle. Later Mishonov [4] obtained an approximate solution to Lamé's problem in the form of divergent triple Fourier series. These series contain constants which are found from infinite systems of linear equations. Teodorescu [5] has considered a particular case of Lamé's problem. Using his own method the author solves the problem in the form of double series analogous to those used in [6 to 8] and by Baida in [9 and 10] in solving problems on the equilibrium of a rectangular parallelepiped. The solution of the problem reduces to three infinite system of linear equations and the author asserts that these infinite systems are regular. It is shown in Section 5 that the infinite systems obtained by Teodorescu, on the other hand, will not be regular.

In the references mentioned above which investigate Lamé's problem the authors confine their attention either to obtaining a solution by an approximate method, or to reducing the solution process to one of obtaining infinite systems, leaving these uninvestigated. It must be emphasized that the main difficulty in solving this problem lies in investigating the infinite systems obtained which are significantly different from the infinite systems of the corresponding plane problem.

In this paper a solution is given to the first fundamental problem of the theory of elasticity for a rectangular parallelepiped with prescribed external stresses on the surface (Sections 2, 3 and 4). For the solution of this problem the author has used a form of the general solution of the homogeneous Lamé equations which contains five arbitrary harmonic functions and which constitutes a generalization of the familiar Papkovich-Neuber solution (Section 1). The solution is expressed in the form of double series containing four series of unknown constants which can be found from four infinite systems of linear algebraic equations. The infinite systems of linear equations obtained is studied for values of Poisson's ratio within the range $0 < \sigma \leq 0.18$. It is shown that for these values of Poisson's ratio the infinite systems are quasi-fully regular.

1. In the absence of body forces the equations of statics for an elastic body are

$$\frac{1}{1-2\sigma}\frac{\partial\theta}{\partial x}+\Delta u=0,\qquad \frac{1}{1-2\sigma}\frac{\partial\theta}{\partial y}+\Delta v=0,\qquad \frac{1}{1-2\sigma}\frac{\partial\theta}{\partial z}+\Delta w=0 \quad (1.1)$$

$$\left(\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

where σ is Poisson's ratio, θ is the volumetic strain and Δ is the Laplace operator.

We seek purely harmonic solutions of Equations (1.1) in the form

$$u=\delta_4(x,y,z),\qquad v=\delta_5(x,y,z),\qquad w=\delta_6(x,y,z) \quad (1.2)$$

where δ_4, δ_5 and δ_6 are harmonic functions. If they satisfy relationship

$$\frac{\partial\delta_4}{\partial x}+\frac{\partial\delta_5}{\partial y}+\frac{\partial\delta_6}{\partial z}=0 \quad (1.3)$$

Equations (1.1) are obviously satisfied.

Adding to the solution (1.2) the biharmonic part of the Papkovich-Neuber solution, we obtain the following form for the general solution of Equations (1.1):

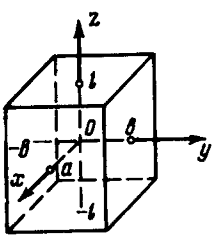

Fig. 1

$$u=\delta_4+\delta_1-\frac{1}{4(1-\sigma)}\frac{\partial}{\partial x}(x\delta_1+y\delta_2+z\delta_3)$$
$$v=\delta_5+\delta_2-\frac{1}{4(1-\sigma)}\frac{\partial}{\partial y}(x\delta_1+y\delta_2+z\delta_3) \quad (1.4)$$
$$w=\delta_6+\delta_3-\frac{1}{4(1-\sigma)}\frac{\partial}{\partial z}(x\delta_1+y\delta_2+z\delta_3)$$

where δ_1, δ_2, δ_3, δ_4, δ_5 and δ_6 are harmonic functions related by (1.3). Setting in (1.4)

$$\delta_4=\frac{\partial\delta_0}{\partial x},\qquad \delta_5=\frac{\partial\delta_0}{\partial y},\qquad \delta_6=\frac{\partial\delta_0}{\partial z}$$

where $\delta_0(x,y,z)$ is a harmonic function, we obtain the Papkovich-Neuber solution.

In the Papkovich-Neuber solution, of the four arbitrary functions only one yields a purely harmonic solution which makes the construction of these solutions difficult. This defect has been pointed out by Hata [11]. Of the five arbitrary harmonic functions in (1.4) two give purely harmonic solutions. This considerably simplifies the selection of the solutions indicated for Equations (1.1) and at the same time facilitates the solution of boundary-value problems for a parallelepiped, which is emphasized in the solution of Lamé's problem outlined below.

2. Without restricting the generality, we shall describe the method of solution for a particular case: we shall consider the deformation of a parallelepiped which is symmetrical about the coordinate planes $x=0$ and $y=0$ (see Fig.1), which occurs in compression and tension.

Thus we require to find the functions $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ which within the parallelepiped $-a\le x\le a$, $-b\le y\le b$, $-l\le z\le l$ satisfy the differential equations (1.1) and on the surface the conditions

$$2G\left(\frac{\partial w}{\partial z}+\frac{\sigma}{1-2\sigma}\theta\right)=\psi_1(x,y)\quad \text{at } z=l \quad (2.1)$$

$$2G\left(\frac{\partial w}{\partial z}+\frac{\sigma}{1-2\sigma}\theta\right)=\psi_2(x,y)\quad \text{at } z=-l \quad (2.2)$$

$$G\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)=f_1(x, y), \qquad G\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=F_1(x, y) \quad \text{at} \quad z=l \tag{2.3}$$

$$G\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)=f_2(x, y), \qquad G\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=F_2(x, y) \quad \text{at} \quad z=-l \tag{2.4}$$

$$2G\left(\frac{\partial u}{\partial x}+\frac{\sigma}{1-2\sigma}\theta\right)=-q(y, z) \quad \text{at} \quad x=\pm a \tag{2.5}$$

$$G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0, \qquad G\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)=0 \quad \text{at} \quad x=\pm a \tag{2.6}$$

$$2G\left(\frac{\partial v}{\partial y}+\frac{\sigma}{1-2\sigma}\theta\right)=-\varphi(x, z) \quad \text{at} \quad y=\pm b \tag{2.7}$$

$$G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0, \qquad G\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)=0 \quad \text{at} \quad y=\pm b \tag{2.8}$$

Here G is the shear modulus. The meaning of the boundary conditions is obvious from the formulas for Hooke's law. We assume that the boundary functions can be represented in Fourier series

$$\psi_i(x, y)=\sum_{p,\, m=0}^{\infty}\lambda_{pm}\psi_{pm}^{(i)}\cos\frac{p\pi x}{a}\cos\frac{m\pi y}{b} \qquad \begin{pmatrix}-a\leqslant x\leqslant a\\ -b\leqslant y\leqslant b\end{pmatrix} \qquad (i=1,2) \tag{2.9}$$

$$f_i(x, y)=\sum_{p,\, m=0}^{\infty}\lambda_{pm}f_{pm}^{(i)}\sin\frac{p\pi x}{a}\cos\frac{m\pi y}{b} \qquad \begin{pmatrix}-a\leqslant x\leqslant a\\ -b\leqslant y\leqslant b\end{pmatrix} \qquad (i=1,2)$$

$$F_i(x, y)=\sum_{p,\, m=0}^{\infty}\lambda_{pm}F_{pm}^{(i)}\cos\frac{p\pi x}{a}\sin\frac{m\pi y}{b} \qquad \begin{pmatrix}-a\leqslant x\leqslant a\\ -b\leqslant y\leqslant b\end{pmatrix} \qquad (i=1,2)$$

$$q(y, z)=\sum_{m,\, n=0}^{\infty}\lambda_{mn}q_{mn}\cos\frac{m\pi y}{b}\cos\frac{n\pi(z-l)}{2l} \qquad \begin{pmatrix}-b\leqslant y\leqslant b\\ -l\leqslant z\leqslant l\end{pmatrix}$$

$$\varphi(x, z)=\sum_{n,\, p=0}^{\infty}\lambda_{np}\varphi_{np}\cos\frac{p\pi x}{a}\cos\frac{n\pi(z-l)}{2l} \qquad \begin{pmatrix}-a\leqslant x\leqslant a\\ -l\leqslant z\leqslant l\end{pmatrix}$$

$$\lambda_{ij}=\begin{cases}1/4 & \text{for} \quad i=j=0\\ 1/2 & \text{for} \quad i=0,\ j>0;\ j=0,\ i>0\\ 1 & \text{for} \quad i>0,\ j>0\end{cases} \tag{2.10}$$

We write the equilibrium condition for the external stresses applied to the parallelepiped as follows

$$\int_{-a}^{a}\int_{-b}^{b}\psi_1(x, y)\,dx\,dy=\int_{-a}^{a}\int_{-b}^{b}\psi_2(x, y)\,dx\,dy=+P$$

where P is the given value of the projection of the resultant on the face $z = l$. Substituting expressions (2.9) for $\psi_i(x, y)$, we find that their Fourier coefficients are related by the expression

$$\psi_{00}^{(1)}=\psi_{00}^{(2)}=+\frac{P}{ab} \tag{2.11}$$

We write the familiar expression for a harmonic function

$$\delta = (C_{11} \cos a_p x + C_{12} \sin \alpha_p x)\, (C_{13} \cos \beta_m y + C_{14} \sin \beta_m y)\, (C_{15} \cosh \gamma_{pm} z + C_{16} \sinh \gamma_{pm} z) + (C_{21} \cos \alpha_p x + C_{22} \sin \alpha_p x)\, (C_{23} \cosh \beta_{np} y + C_{24} \sinh \beta_{np}\, y) \times [C_{25} \cos \gamma_n (z - l) + C_{26} \sin \gamma_n (z - l)] + (C_{31} \cosh \alpha_{mn} x + C_{32} \sinh \alpha_{mn} x) \times (C_{33} \cos \beta_m y + C_{34} \sin \beta_m y)\, [C_{35} \cos \gamma_n (z - l) + C_{36} \sin \gamma_n (z - l)]] \tag{2.12}$$

$$\alpha_p = \frac{p\pi}{a}, \qquad \beta_m = \frac{m\pi}{b}, \qquad \gamma_n = \frac{n\pi}{2l} \tag{2.13}$$

$$\alpha_{mn} = \sqrt{\beta_m{}^2 + \gamma_n{}^2}, \qquad \beta_{np} = \sqrt{\gamma_n{}^2 + \alpha_p{}^2}, \qquad \gamma_{pm} = \sqrt{\alpha_p{}^2 + \beta_m{}^2}$$

Here $C_{11}, C_{12}, \ldots, C_{36}$ are arbitrary constants.

In deriving a solution to the boundary-value problem, we shall use (2.12) and select expressions for the harmonic functions appearing in (1.4) such that the series composed of the particular solutions (1.4) satisfy the boundary conditions (2.3), (2.4), (2.6) and (2.8) in the directions tangential to the surface of the parallelepiped. This is achieved by selecting the constants in (2.12) and bearing in mind the evenness and oddness of the appropriate displacements and stresses. Thus setting in (1.4)

$$\delta_1 = 0, \qquad \delta_2 = 0, \qquad \delta_3 = \frac{1}{l\gamma_{pm}{}^2 \sinh \gamma_{pm} l}\, [C_{pm}^{(3)} \sinh \gamma_{pm} z + C_{pm}^{(4)} \cosh \gamma_{pm} z] \times \cos \alpha_p x \cos \beta_m y$$

$$\delta_4 = -\alpha_p\, [C_{pm}^{(3)} M_{pm}^{(3)} \cosh \gamma_{pm} z + C_{pm}^{(4)} M_{pm}^{(4)} \sinh \gamma_{pm} z] \sin \alpha_p x \cos \beta_m y + \frac{\lambda_{pm}}{4G} \Big\{ [(2\gamma_{pm}{}^2 - \alpha_p{}^2)(f_{pm}^{(1)} - f_{pm}^{(2)}) - \beta_m \alpha_p (F_{pm}^{(1)} - F_{pm}^{(2)})] \frac{\cosh \gamma_{pm} z}{\gamma_{pm}{}^3 \sinh \gamma_{pm} l} + [(2\gamma_{pm}{}^2 - \alpha_p{}^2)(f_{pm}^{(1)} + f_{pm}^{(2)}) - \beta_m \alpha_p (F_{pm}^{(1)} + F_{pm}^{(2)})] \frac{\sinh \gamma_{pm} z}{\gamma_{pm}{}^3 \cosh \gamma_{pm} l} \Big\} \sin \alpha_p x \cos \beta_m y$$

$$\delta_5 = -\beta_m\, [C_{pm}^{(3)} M_{pm}^{(3)} \cosh \gamma_{pm} z + C_{pm}^{(4)} M_{pm}^{(4)} \sinh \gamma_{pm} z] \cos \alpha_p x \sin \beta_m y + \frac{\lambda_{pm}}{4G} \Big\{ [(2\gamma_{pm}{}^2 - \beta_m{}^2)(F_{pm}^{(1)} - F_{pm}^{(2)}) - \alpha_p \beta_m (f_{pm}^{(1)} - f_{pm}^{(2)})] \frac{\cosh \gamma_{pm} z}{\gamma_{pm}{}^3 \sinh \gamma_{pm} l} + [(2\gamma_{pm}{}^2 - \beta_m{}^2)(F_{pm}^{(1)} + F_{pm}^{(2)}) - \alpha_p \beta_m (f_{pm}^{(1)} + f_{pm}^{(2)})] \frac{\sinh \gamma_{pm} z}{\gamma_{pm}{}^3 \cosh \gamma_{pm} l} \Big\} \cos \alpha_p x \sin \beta_m y$$

$$\delta_6 = \gamma_{pm}\, [C_{pm}^{(3)} M_{pm}^{(3)} \sinh \gamma_{pm} z + C_{pm}^{(4)} M_{pm}^{(4)} \cosh \gamma_{pm} z] \cos \alpha_p x \cos \beta_m y - \frac{\lambda_{pm}}{4C} \Big\{ [\alpha_p (f_{pm}^{(1)} - f_{pm}^{(2)}) + \beta_m (F_{pm}^{(1)} - F_{pm}^{(2)})] \frac{\sinh \gamma_{pm} z}{\gamma_{pm}{}^2 \sinh \gamma_{pm} l} + [\alpha_p (f_{pm}^{(1)} + f_{pm}^{(2)}) + \beta_m (F_{pm}^{(1)} + F_{pm}^{(2)})] \frac{\cosh \gamma_{pm} z}{\gamma_{pm}{}^2 \cosh \gamma_{pm} l} \Big\} \cos \alpha_p x \cos \beta_m y$$

$$M_{pm}^{(3)} = \frac{2\sigma - 1 + \gamma_{pm} l \coth \gamma_{pm} l}{4(1-\sigma)\, l\gamma_{pm}{}^3 \sinh \gamma_{pm} l}, \qquad M_{pm}^{(4)} = \frac{2\sigma - 1 + \gamma_{pm} l \tanh \gamma_{pm} l}{4(1-\sigma)\, l\gamma_{pm}{}^3 \sinh \gamma_{pm} l}$$

we obtain the following particular solution to the Lamé equations (1.1):

$$u_{pm}^{(3)} = \frac{\alpha_p}{4(1-\sigma)\, l\gamma_{pm}^3 \sinh\gamma_{pm} l}\{C_{pm}^{(3)}\ [(1-2\sigma-\gamma_{pm}l\ \coth\gamma_{pm}l)\cosh\gamma_{pm}z+\gamma_{pm}z\ \times$$

$$\times \sinh\gamma_{pm}z]+C_{pm}^{(4)}[(1-2\sigma-\gamma_{pm}l\tanh\gamma_{pm}l)\sinh\gamma_{pm}z+\gamma_{pm}z\cosh\gamma_{pm}z]\}\sin\frac{p\pi x}{a}\cos\frac{m\pi y}{b}+$$

$$+\frac{\lambda_{pm}}{4G\gamma_{pm}^3}\left\{[(2\gamma_{pm}^2-\alpha_p^2)(f_{pm}^{(1)}-f_{pm}^{(2)})-\beta_m\alpha_p(F_{pm}^{(1)}-F_{pm}^{(2)})]\frac{\cosh\gamma_{pm}z}{\sinh\gamma_{pm}l}+\right.$$

$$\left.+[(2\gamma_{pm}^2-\alpha_p^2)(f_{pm}^{(1)}+f_{pm}^{(2)})-\beta_m\alpha_p(F_{pm}^{(1)}+F_{pm}^{(2)})]\frac{\sinh\gamma_{pm}z}{\cosh\gamma_{pm}l}\right\}\sin\frac{p\pi x}{a}\cos\frac{m\pi y}{b}$$

$$v_{pm}^{(3)} = \frac{\beta_m}{4(l-\sigma)\, l\gamma_{pm}^3\sinh\gamma_{pm}l}\{C_{pm}^{(3)}[(1-2\sigma-\gamma_{pm}l\ \coth\gamma_{pm}l)\cosh\gamma_{pm}z\ +$$

$$+\gamma_{pm}z\sinh\gamma_{pm}z]+C_{pm}^{(4)}[(1-2\sigma-\gamma_{pm}l\tanh\gamma_{pm}l)\sinh\gamma_{pm}z+ \qquad (2.14)$$

$$+\gamma_{pm}z\cosh\gamma_{pm}z]\}\cos\frac{p\pi x}{a}\sin\frac{m\pi y}{b}+\frac{\lambda_{pm}}{4G\gamma_{pm}^3}\times$$

$$\times\left\{[(2\gamma_{pm}^2-\beta_m^2)(F_{pm}^{(1)}-F_{pm}^{(2)})-\alpha_p\beta_m(f_{pm}^{(1)}-f_{pm}^{(2)})]\frac{\cosh\gamma_{pm}z}{\sinh\gamma_{pm}l}+\right.$$

$$\left.+[(2\gamma_{pm}^2-\beta_m^2)(F_{pm}^{(1)}+F_{pm}^{(2)})-\alpha_p\beta_m(f_{pm}^{(1)}+f_{pm}^{(2)})]\frac{\sinh\gamma_{pm}z}{\cosh\gamma_{pm}l}\right\}\cos\frac{p\pi x}{a}\sin\frac{m\pi y}{b}$$

$$w_{pm}^{(3)} = \frac{1}{4(1-\sigma)\, l\gamma_{pm}^2\sinh\gamma_{pm}l}\{C_{pm}^{(3)}[(2-2\sigma+\gamma_{pm}l\ \coth\gamma_{pm}l)\sinh\gamma_{pm}z\ -$$

$$-\gamma_{pm}z\cosh\gamma_{pm}z]+C_{pm}^{(4)}[(2-2\sigma+\gamma_{pm}l\tanh\gamma_{pm}l)\cosh\gamma_{pm}z-\gamma_{pm}z\sinh\gamma_{pm}z]\}\times$$

$$\times\cos\frac{p\pi x}{a}\cos\frac{m\pi y}{b}-\frac{\lambda_{pm}}{4G\gamma_{mp}^2}\left\{[\alpha_p(f_{pm}^{(1)}-f_{pm}^{(2)})+\beta_m(F_{pm}^{(1)}-F_{pm}^{(2)})]\frac{\sinh\gamma_{pm}z}{\sinh\gamma_{pm}l}+\right.$$

$$\left.+[\alpha_p(f_{pm}^{(1)}+f_{pm}^{(2)})+\beta_m(F_{pm}^{(1)}+F_{pm}^{(2)})]\frac{\cosh\gamma_{pm}z}{\cosh\gamma_{pm}l}\right\}\cos\frac{p\pi x}{a}\cos\frac{m\pi y}{b}$$

$$(p, m = 0, 1, 2, \ldots, \gamma_{pm}\neq 0)$$

Here $C_{pm}^{(3)}$ and $C_{pm}^{(4)}$ are arbitrary constants, $F_{pm}^{(i)}$ and $f_{pm}^{(i)}$ are the Fourier coefficients of series (2.9).

In expressions (2.14) we select the particular solutions which contain only the constant $C_{pm}^{(3)}$ and carry out a cyclic permutation of variables and parameters. As a result we obtain two further types of particular solutions to Lamé's equations

$$u_{mn}^{(1)} = \frac{C_{mn}^{(1)}}{4(1-\sigma)\, a\alpha_{mn}^2\sinh a\alpha_{mn}}[(2-2\sigma+\alpha_{mn}a\ \coth\alpha_{mn}a)\sinh\alpha_{mn}x\ -$$

$$-\alpha_{mn}x\cosh\alpha_{mn}x]\cos\frac{m\pi y}{b}\cos\frac{n\pi(z-l)}{2l}$$

$$v_{mn}^{(1)} = \frac{\beta_m C_{mn}^{(1)}}{4(1-\sigma)\, a\alpha_{mn}^3 \sinh a\alpha_{mn}} [(1 - 2\sigma - \alpha_{mn} a \coth \alpha_{mn} a) \cosh \alpha_{mn} x + \alpha_{mn} x \sinh \alpha_{mn} x] \sin \frac{m\pi y}{b} \cos \frac{n\pi(z-l)}{2l} \tag{2.15}$$

$$w_{mn}^{(1)} = \frac{\gamma_n C_{mn}^{(1)}}{4(1-\sigma)\, a\alpha_{mn}^3 \sinh a\alpha_{mn}} [(1 - 2\sigma - \delta\alpha_{mn} \coth a\alpha_{mn}) \cosh \alpha_{mn} x + \alpha_{mn} x \sinh \alpha_{mn} x] \cos \frac{m\pi y}{b} \sin \frac{n\pi(z-l)}{2l}$$

$$u_{np}^{(2)} = \frac{\alpha_p C_{np}^{(2)}}{4(1-\sigma)\, b\beta_{np}^3 \sinh b\beta_{np}} [(1 - 2\sigma - b\beta_{np} \coth b\beta_{np}) \cosh \beta_{np} y + \beta_{np} y \sinh \beta_{np} y] \sin \frac{p\pi x}{a} \cos \frac{n\pi(z-l)}{2l}$$

$$v_{np}^{(2)} = \frac{C_{np}^{(2)}}{4(1-\sigma)\, b\beta_{np}^2 \sinh b\beta_{np}} [(2 - 2\sigma + b\beta_{np} \coth b\beta_{np}) \sinh \beta_{np} y - \beta_{np} y \cosh \beta_{np} y] \cos \frac{p\pi x}{a} \cos \frac{n\pi(z-l)}{2l} \tag{2.16}$$

$$w_{np}^{(2)} = \frac{\gamma_n C_{np}^{(2)}}{4(1-\sigma)\, b\beta_{np}^3 \sinh b\beta_{np}} [(1 - 2\sigma - b\beta_{np} \coth b\beta_{np}) \cosh \beta_{np} y + \beta_{np} y \sinh \beta_{np} y] \cos \frac{p\pi x}{a} \sin \frac{n\pi(z-l)}{2l}$$

$$(m, n, p = 0, 1, 2, \ldots, \beta_{np} \neq 0, \alpha_{mn} \neq 0)$$

where $C_{mn}^{(1)}$ and $C_{pm}^{(4)}$ are arbitrary constants.

We seek a solution in the form of the double series

$$u = \frac{1-2\sigma}{4(1-\sigma)} C_0^{(1)} x + \sum_{m,n=0}^{\infty}{}' u_{mn}^{(1)} + \sum_{n,p=0}^{\infty}{}' u_{np}^{(2)} + \sum_{p,m=0}^{\infty}{}' u_{pm}^{(3)}$$

$$v = \frac{1-2\sigma}{4(1-\sigma)} C_0^{(2)} y + \sum_{m,n=0}^{\infty}{}' v_{mn}^{(1)} + \sum_{n,p=0}^{\infty}{}' v_{np}^{(2)} + \sum_{p,m=0}^{\infty}{}' v_{pm}^{(3)} \tag{2.17}$$

$$w = \frac{1-2\sigma}{4(1-\sigma)} C_0^{(3)} z + \sum_{m,n=0}^{\infty}{}' w_{mn}^{(1)} + \sum_{n,p=0}^{\infty}{}' w_{np}^{(2)} + \sum_{p,m=0}^{\infty}{}' w_{pm}^{(3)}$$

where $C_0^{(1)}$, $C_0^{(2)}$ and $C_0^{(3)}$ are arbitrary constants. The dash above the summation signs indicates that the summution indices are not zero simultaneously. The series (2.17), with no indication of the method of derivation, were given in [6] in the solution of a mixed problem on the compression of a rectangular parallelepiped.

Differentiation confirms that the functions represented by the series (2.17) satisfy the boundary conditions (2.3),(2.4), (2.6) and (2.8) for shear stresses. The unknown constants $C_0^{(1)}, C_0^{(2)}, \ldots, C_{pm}^{(4)}$, in (2.17) can be determined uniquely from the boundary conditions (2.1),(2.2),(2.5) and (2.7)

After satisfying the boundary conditions (2.1),(2.2),(2.5) and (2.7) we obtain the four equalities

$$\frac{\sigma G}{2(1-\sigma)}\left[C_0^{(1)}+C_0^{(2)}+C_0^{(3)}\frac{1-\sigma}{\sigma}\right]+\sum_{m,\,n=0}^{\infty}{}' \Phi_5(x,m,n)\cos\frac{m\pi y}{b}+$$
$$+\sum_{n,\,p=0}^{\infty}{}' \Phi_6(y,n,p)\cos\frac{p\pi x}{a}+\sum_{p,\,m=0}^{\infty}{}'\left\{\frac{G}{2(1-\sigma)}\left[L_{pm}^{(3)}\frac{C_{pm}^{(3)}}{\gamma_{pm}l}+R_{pm}^{(4)}\frac{C_{pm}^{(4)}}{l\gamma_{pm}}\right]+\right.$$
$$\left.+\,\omega_{pm}^{(1)}\right\}\cos\frac{p\pi x}{a}\cos\frac{m\pi y}{b}=\sum_{p,\,m=0}^{\infty}\lambda_{pm}\psi_{pm}^{(1)}\cos\frac{p\pi x}{a}\cos\frac{m\pi y}{b}$$

$$\frac{\sigma G}{2(1-\sigma)}\left[C_0^{(1)}+C_0^{(2)}+\frac{1-\sigma}{\sigma}C_0^{(3)}\right]+\sum_{m,\,n=0}^{\infty}{}'(-1)^n\Phi_5(x,m,n)\cos\frac{m\pi y}{b}+$$
$$+\sum_{n,\,p=0}^{\infty}{}'(-1)^n\Phi_6(y,n,p)\cos\frac{p\pi x}{a}+\sum_{p,\,m=0}^{\infty}{}'\left\{\frac{G}{2(1-\sigma)l\gamma_{pm}}\left[L_{pm}^{(3)}C_{pm}^{(3)}-\right.\right.$$
$$\left.\left.-\,R_{pm}^{(4)}C_{pm}^{(4)}\right]+\omega_{pm}^{(2)}\right\}\cos\frac{p\pi x}{a}\cos\frac{m\pi y}{b}=\sum_{p,\,m=0}^{\infty}\lambda_{pm}\psi_{pm}^{(2)}\cos\frac{p\pi x}{a}\cos\frac{m\pi y}{b}\qquad(2.18)$$

$$\frac{\sigma G}{2(1-\sigma)}\left[\frac{1-\sigma}{\sigma}C_0^{(1)}+C_0^{(2)}+C_0^{(3)}\right]+\frac{G}{2(1-\sigma)}\sum_{m,\,n=0}^{\infty}{}'\frac{C_{mn}^{(1)}L_{mn}^{(1)}}{a\alpha_{mn}}\cos\frac{m\pi y}{b}\times$$
$$\times\cos\frac{n\pi(z-l)}{2l}+\sum_{n,\,p=0}^{\infty}{}'\Phi_1(y,n,p)\cos\frac{n\pi(z-l)}{2l}+$$
$$+\sum_{p,\,m=0}^{\infty}{}'\Phi_2(z,p,m)\cos\frac{m\pi y}{b}=-\sum_{m,\,n=0}^{\infty}\lambda_{mn}q_{mn}\cos\frac{m\pi y}{b}\cos\frac{n\pi(z-l)}{2l}$$

$$\frac{\sigma G}{2(1-\sigma)}\left[C_0^{(1)}+\frac{1-\sigma}{\sigma}C_0^{(2)}+C_0^{(3)}\right]+\frac{G}{2(1-\sigma)}\sum_{n,\,p=0}^{\infty}\frac{C_{np}^{(2)}L_{np}^{(2)}}{b\beta_{np}}\cos\frac{p\pi x}{a}\times$$
$$\times\cos\frac{n\pi(z-l)}{2l}+\sum_{m,\,n=0}^{\infty}{}'\Phi_3(x,m,n)\cos\frac{n\pi(z-l)}{2l}+$$
$$+\sum_{p,\,m=0}^{\infty}{}'\Phi_4(z,p,m)\cos\frac{p\pi x}{a}=-\sum_{p,\,n=0}^{\infty}\lambda_{np}\varphi_{np}\cos\frac{p\pi x}{a}\cos\frac{n\pi(z-l)}{2l}$$

in which we have introduced the notations

$$L_{mn}^{(1)}=\coth a\alpha_{mn}+\frac{\alpha_{mn}a}{\sinh^2\alpha_{mn}a},\qquad L_{np}^{(2)}=\coth\beta_{np}b+\frac{\beta_{np}b}{\sinh^2\beta_{np}b}\qquad(2.19)$$
$$L_{pm}^{(3)}=\coth\gamma_{pm}l+\frac{\gamma_{pm}l}{\sinh^2\gamma_{pm}l},\qquad R_{pm}^{(4)}=1-\frac{2\gamma_{pm}l}{\sinh 2\gamma_{pm}l}$$

$$\omega_{pm}^{(1)}=-\frac{\lambda_{pm}}{2\gamma_{pm}}\{[\alpha_p(f_{pm}^{(1)}-f_{pm}^{(2)})+\beta_m(F_{pm}^{(1)}-F_{pm}^{(2)})]\coth\gamma_{pm}l+$$
$$+[\alpha_p(f_{pm}^{(1)}+f_{pm}^{(2)})+\beta_m(F_{pm}^{(1)}+F_{pm}^{(2)})\tanh\gamma_{pm}l]\}\qquad(2.20)$$

$$\omega_{pm}^{(2)} = -\frac{\lambda_{pm}}{2\gamma_{pm}} \{[\alpha_p (f_{pm}^{(1)} - f_{pm}^{(2)}) + \beta_m (F_{pm}^{(1)} - F_{pm}^{(2)})] \coth \gamma_{pm} l -$$
$$- [\alpha_p (f_{pm}^{(1)} + f_{pm}^{(2)}) + \beta_m (F_{pm}^{(1)} + F_{pm}^{(2)})] \tanh \gamma_{pm} l\}$$

$$\Phi_1 (y, n, p) = \frac{G(-1)^p C_{np}^{(2)}}{2(1-\sigma) b\beta_{np}^3 \sinh b\beta_{np}} \{[(1-2\sigma)\alpha_p^2 + 2\sigma\beta_{np}^2 -$$
$$- \alpha_p^2 b\beta_{np} \coth b\beta_{np}] \cosh \beta_{np} y + \alpha_p^2 \beta_{np} y \sinh \beta_{np} y\}$$

$$\Phi_2 (z, p, m) = \frac{G(-1)^p C_{pm}^{(3)}}{2(1-\sigma) l\gamma_{pm}^3 \sinh \gamma_{pm} l} \{[(1-2\sigma)\alpha_p^2 + 2\sigma\gamma_{pm}^2 -$$
$$- \alpha_p^2 \gamma_{pm} l \coth \gamma_{pm} l] \cosh \gamma_{pm} z + \alpha_p^2 \gamma_{pm} z \sinh \gamma_{pm} z\} +$$
$$+ \frac{G(-1)^p C_{pm}^{(4)}}{2(1-\sigma) l\gamma_{pm}^3 \sinh \gamma_{pm} l} \{[(1-2\sigma)\alpha_p^2 + 2\sigma\gamma_{pm}^2 - \qquad (2.21)$$
$$- \alpha_p^2 l\gamma_{pm} \tanh \gamma_{pm} l] \sinh \gamma_{pm} z + \alpha_p^2 \gamma_{pm} z \cosh \gamma_{pm} z\} +$$
$$+ \frac{(-1)^p \lambda_{pm}}{2\gamma_{pm}^3} \{[\alpha_p (2\gamma_{pm}^2 - \alpha_p^2)(f_{pm}^{(1)} - f_{pm}^{(2)}) - \beta_m \alpha_p^2 (F_{pm}^{(1)} - F_{pm}^{(2)})] \frac{\cosh \gamma_{pm} z}{\sinh \gamma_{pm} l} +$$
$$+ [\alpha_p (2\gamma_{pm}^2 - \alpha_p^2)(f_{pm}^{(1)} + f_{pm}^{(2)}) - \beta_m \alpha_p^2 (F_{pm}^{(1)} + F_{pm}^{(2)})] \frac{\sinh \gamma_{pm} z}{\cosh \gamma_{pm} l}\}$$

$$\Phi_3 (x, m, n) = \frac{G(-1)^m C_{mn}^{(1)}}{2(1-\sigma) a\alpha_{mn}^3 \sinh a\alpha_{mn}} \{\beta_m^2 \alpha_{mn} x \sinh \alpha_{mn} x +$$
$$+ [(1-2\sigma)\beta_m^2 + 2\sigma\alpha_{mn}^2 - \beta_m^2 a\alpha_{mn} \coth a\alpha_{mn}] \cosh \alpha_{mn} x\}$$

$$\Phi_4 (z, p, m) = \frac{G(-1)^m}{2(1-\sigma) l\gamma_{pm}^3 \sinh \gamma_{pm} l} \{C_{pm}^{(3)} [(2\sigma\gamma_{pm}^2 + (1-2\sigma)\beta_m^2 -$$
$$- \beta_m^2 \gamma_{pm} l \coth \gamma_{pm} l) \cosh \gamma_{pm} z + \beta_m^2 \gamma_{pm} z \sinh \gamma_{pm} z] +$$
$$+ C_{pm}^{(4)} [(2\sigma\gamma_{pm}^2 + (1-2\sigma)\beta_m^2 - \beta_m^2 \gamma_{pm} l \tanh \gamma_{pm} l) \sinh \gamma_{pm} z + \beta_m^2 \gamma_{pm} z \cosh \gamma_{pm} z]\} +$$
$$+ \frac{(-1)^m \beta_m \lambda_{pm}}{2\gamma_{pm}^3} \left\{[(2\gamma_{pm}^2 - \beta_m^2)(F_{pm}^{(1)} - F_{pm}^{(2)}) - \beta_m \alpha_p (f_{pm}^{(1)} - f_{pm}^{(2)})] \frac{\cosh \gamma_{pm} z}{\sinh \gamma_{pm} l} +\right.$$
$$\left. + [(2\gamma_{pm}^2 - \beta_m^2)(F_{pm}^{(1)} + F_{pm}^{(2)}) - \alpha_p \beta_m (f_{pm}^{(1)} + f_{pm}^{(2)})] \frac{\sinh \gamma_{pm} z}{\cosh \gamma_{pm} l}\right\}$$

$$\Phi_5 (x, m, n) = \frac{GC_{mn}^{(1)}}{2(1-\sigma) a\alpha_{mn}^3 \sinh a\alpha_{mn}} \{\gamma_n^2 \alpha_{mn} x \sinh \alpha_{mn} x +$$
$$+ [(1-2\sigma)\gamma_n^2 + 2\sigma\alpha_{mn}^2 - \gamma_n^2 a\alpha_{mn} \coth \alpha_{mn} a] \cosh \alpha_{mn} x\}$$

$$\Phi_6 (y, n, p) = \frac{GC_{np}^{(2)}}{2(1-\sigma) b\beta_{np}^3 \sinh b\beta_{np}} \{\gamma_n^2 \beta_{np} y \sinh \beta_{np} y +$$
$$+ [(1-2\sigma)\gamma_n^2 + 2\sigma\beta_{np}^2 - \gamma_n^2 b\beta_{np} \coth b\beta_{np}] \cosh \beta_{np} y\}$$

To find the unknown constants $C_0^{(1)}, \ldots, C_{pm}^{(4)}$, we equate the Fourier coefficients of the functions in the left- and right-hand sides of (2.18). For this purpose we expand the functions (2.21) into the appropriate Fourier series and substitute their expressions into (2.18). By equating the Fourier coefficients for zero values of both indices we obtain three equations in the constants $C_0^{(1)}$, $C_0^{(2)}$ and $C_0^{(3)}$

$$C_0^{(1)} + C_0^{(2)} + \frac{1-\sigma}{\sigma} C_0^{(3)} = \frac{(1-\sigma)\,\psi_{00}^{(1)}}{2\sigma G}$$

$$\frac{1-\sigma}{\sigma} C_0^{(1)} + C_0^{(2)} + C_0^{(3)} = \frac{1-\sigma}{\sigma G}\left[-\frac{q_{00}}{2} - \sum_{s=1}^{\infty} \frac{a(-1)^s (f_{s0}^{(1)} - f_{s0}^{(2)})}{2\pi l s}\right] \tag{2.22}$$

$$C_0^{(1)} + \frac{1-\sigma}{\sigma} C_0^{(2)} + C_0^{(3)} = \frac{1-\sigma}{\sigma G}\left[-\frac{\varphi_{00}}{2} - \sum_{s=1}^{\infty} \frac{b(-1)^s (F_{0s}^{(1)} - F_{0s}^{(2)})}{2\pi l s}\right]$$

and from the first two equalities (2.18), by virtue of (2.11), we obtain the first equation of (2.22). In (2.18), equating Fourier coefficients for values of the indices not simultaneously zero, we obtain four relations which after some transformations assume the form

$$A_{mn}^{(1)} = -\sum_{p=0}^{\infty} H_{mnp}^{(11)} A_{np}^{(2)} -$$

$$-\frac{1}{2}(-1)^n \sum_{s=0}^{\infty} H_{nsm}^{(12)} \{[1+(-1)^n] A_{sm}^{(3)} + [1-(-1)^n] A_{sm}^{(4)}\} + b_{mn}^{(1)} \tag{2.23}$$

$$(m,\ n = 0, 1, \ldots;\ m \neq 0 \text{ and } p \neq 0 \text{ when } n = 0;\ s \neq 0 \text{ and } n \neq 0 \text{ when } m = 0)$$

$$A_{np}^{(2)} = -\sum_{m=0}^{\infty} H_{pmn}^{(21)} A_{mn}^{(1)} -$$

$$-\frac{1}{2}(-1)^n \sum_{s=0}^{\infty} H_{nps}^{(22)} \{[1+(-1)^n] A_{ps}^{(3)} + [1-(-1)^n] A_{ps}^{(4)}\} + b_{np}^{(2)} \tag{2.24}$$

$$(n,\ p = 0, 1, 2, \ldots;\ p \neq 0 \text{ and } m \neq 0 \text{ when } n = 0;\ s \neq 0 \text{ and } n \neq 0 \text{ when } p = 0)$$

$$A_{pm}^{(3)} = -\sum_{n=0,\,2,\,\ldots}^{\infty} H_{pmn}^{(31)} A_{mn}^{(1)} - \sum_{s=0,\,2,\,\ldots}^{\infty} H_{msp}^{(32)} A_{sp}^{(2)} + b_{pm}^{(3)} \tag{2.25}$$

$$(p,\ m = 0, 1, 2, \ldots;\ s \neq 0 \text{ and } n \neq 0 \text{ when } p = 0;\ p \neq 0 \text{ and } n \neq 0 \text{ when } m = 0)$$

$$A_{pm}^{(4)} = \sum_{n=1,\,3,\,\ldots}^{\infty} H_{pmn}^{(41)} A_{mn}^{(1)} + \sum_{s=1,\,3,\,\ldots}^{\infty} H_{msp}^{(42)} A_{sp}^{(2)} + b_{pm}^{(4)} \tag{2.26}$$

$$(p,\ m = 0, 1, 2, \ldots;\ p \neq 0 \text{ when } m = 0;\ m \neq 0 \text{ when } p = 0)$$

where

$$H_{mnp}^{(11)} = \frac{4\lambda_m g(\alpha_p, \beta_m, \gamma_n)}{a L_{mn}^{(1)} \beta_{np}}, \qquad H_{nsm}^{(12)} = \frac{4\lambda_n g(\gamma_n, \alpha_s, \beta_m)}{a L_{mn}^{(1)} \gamma_{sm}}$$

$$H_{pmn}^{(21)} = \frac{4\lambda_p g(\alpha_p, \beta_m, \gamma_n)}{b L_{np}^{(2)} \alpha_{mn}}, \qquad H_{nps}^{(22)} = \frac{4\lambda_n g(\beta_s, \gamma_n, \alpha_p)}{b L_{np}^{(2)} \gamma_{ps}}$$

$$H_{pmn}^{(31)} = \frac{4\lambda_p g(\gamma_n, \alpha_p, \beta_m)}{lL_{pm}^{(3)}\alpha_{mn}}, \quad H_{msp}^{(32)} = \frac{4\lambda_m g(\beta_m, \gamma_s, \alpha_p)}{4L_{pm}^{(3)}\beta_{sp}} \tag{2.27}$$

$$H_{pmn}^{(41)} = \frac{4\lambda_p g(\gamma_n, \alpha_p, \beta_m)}{lL_{pm}^{(4)}\alpha_{mn}}, \quad H_{msp}^{(42)} = \frac{4\lambda_m g(\beta_m, \gamma_s, \alpha_p)}{lL_{pm}^{(4)}\beta_{sp}}$$

$$g(\alpha_p, \beta_m, \gamma_n) = \frac{\alpha_p^2 \beta_m^2}{(\alpha_p^2 + \beta_m^2 + \gamma_n^2)^2} + \frac{\sigma\gamma_n^2}{\alpha_p^2 + \beta_m^2 + \gamma_n^2} \tag{2.28}$$

$$L_{pm}^{(4)} = R_{pm}^{(4)} \tanh\gamma_{mp} l = \tanh\gamma_{pm} l - \frac{\gamma_{pm} l}{\cosh^2\gamma_{pm} l}, \qquad L_{pm}^{(4)} > 0$$

$$b_{mn}^{(1)} = \frac{l(-1)^m(-1)^n}{a^2\alpha_{mn}} \xi_{mn}^{(1)}, \qquad b_{np}^{(2)} = \frac{l(-1)^n(-1)^p}{b^2\beta_{np}} \xi_{np}^{(2)} \tag{2.29}$$

$$b_{pm}^{(3)} = \frac{(-1)^p(-1)^m}{\gamma_{pm} l} \xi_{pm}^{(3)}, \quad b_{pm}^{(4)} = \frac{(-1)^p(-1)^m}{\gamma_{pm} l \tanh\gamma_{pm} l} \xi_{pm}^{(4)} \tag{2.30}$$

$$\xi_{mn}^{(1)} = -\frac{a\alpha_{mn}}{L_{mn}^{(1)}} \sum_{s=0}^{\infty} \lambda_n \eta_n^{(1)}(s, m) - \frac{2(1-\sigma)a}{GL_{mn}^{(1)}} \lambda_{mn}\alpha_{mn} q_{mn} \qquad (s \neq 0 \text{ for } m = 0)$$

$$\xi_{np}^{(2)} = -\frac{b\beta_{np}}{L_{np}^{(2)}} \sum_{s=0}^{\infty} \lambda_n \eta_n^{(2)}(p, s) - \frac{2(1-\sigma)b}{GL_{np}^{(2)}} \lambda_{np}\beta_{np}\varphi_{np} \qquad (s \neq 0 \text{ for } p = 0)$$

$$\xi_{pm}^{(3)} = \frac{(1-\sigma)\gamma_{pm} l}{GL_{pm}^{(3)}} [-\omega_{pm}^{(1)} - \omega_{pm}^{(2)} + \lambda_{pm}\psi_{pm}^{(1)} + \lambda_{pm}\psi_{pm}^{(2)}]$$

$$\xi_{pm}^{(4)} = \frac{(1-\sigma)\gamma_{pm} l}{GR_{pm}^{(4)}} [\omega_{pm}^{(2)} - \omega_{pm}^{(1)} + \lambda_{pm}\psi_{pm}^{(1)} - \lambda_{pm}\psi_{pm}^{(2)}]$$

$$\eta_n^{(1)}(s, m) = \frac{2(1-\sigma)(-1)^s \lambda_{sm}\alpha_s}{Gl\gamma_{sm}^2(\gamma_n^2 + \gamma_{sm}^2)} \{(2\gamma_{sm}^2 - \alpha_s^2)[f_{sm}^{(1)} - (-1)^n f_{sm}^{(2)}] - \beta_m\alpha_s[F_{sm}^{(1)} - (-1)^n F_{sm}^{(2)}]\}$$

$$\eta_n^{(2)}(p, s) = \frac{2(1-\sigma)(-1)^s \lambda_{ps}\beta_s}{Gl\gamma_{ps}^2(\gamma_n^2 + \gamma_{ps}^2)} \{(2\gamma_{ps}^2 - \beta_s^2)[F_{ps}^{(1)} - (-1)^n F_{ps}^{(2)}] - \beta_s\alpha_p[f_{ps}^{(1)} - (-1)^n f_{ps}^{(2)}]\}$$

$$\lambda_i = \begin{cases} 1/2, & \text{for } i = 0 \\ 1 & \text{for } i = 1, 2, \ldots \end{cases} \tag{2.31}$$

Here, instead of the unknown constants $C_{mn}^{(1)}, \ldots, C_{pm}^{(4)}$, we have introduced new constants $A_{mn}^{(1)}, \ldots, A_{pm}^{(4)}$ given by Formulas

$$C_{mn}^{(1)} = \frac{a^2}{l}(-1)^m(-1)^n \alpha_{mn} A_{mn}^{(1)}, \qquad C_{np}^{(2)} = \frac{b^2}{l}(-1)^n(-1)^p \beta_{np} A_{np}^{(2)}$$

$$C_{pm}^{(3)} = l(-1)^p(-1)^m \gamma_{pm} A_{pm}^{(3)}, \quad C_{pm}^{(4)} = l(-1)^p(-1)^m \gamma_{pm} A_{pm}^{(4)} \tanh\gamma_{pm} l \tag{2.32}$$

The functions $g(\beta_m, \gamma_n, \alpha_p)$, $g(\gamma_n, \alpha_p, \beta_m)$ can be obtained from the function $g(\alpha_p, \beta_m, \gamma_n)$ by cyclic permutation of arguments.

Thus all boundary conditions are satisfied and the solution of the problem is given by the series (2.17). The constants $C_0^{(1)}$, $C_0^{(2)}$ and $C_0^{(3)}$ in the series (2.17) can be found from (2.22) and the constants $C_{mn}^{(1)}, \ldots, C_{pm}^{(4)}$ can be expressed in terms of the new constants $A_{mn}^{(1)}, \ldots, A_{pm}^{(4)}$ by means of Formulas (2.32). In order to find the latter we have the infinite systems (2.23) to (2.26) of linear algebraic equations.

3. It can easily be seen that by applying the rectangle formula we obtain

$$\sum_{p=1}^{\infty} \frac{\gamma^3}{(\gamma^2+p^2)(\gamma^2+\beta^2+p^2)} < \gamma^3 \int_0^{+\infty} \frac{dx}{(\gamma^2+x^2)(\gamma^2+\beta^2+x^2)} \qquad (\gamma > 0)$$

or, after evaluating the integral,

$$\sum_{p=1}^{\infty} \frac{\gamma^3}{(\gamma^2+p^2)(\gamma^2+\beta^2+p^2)} < \frac{\pi}{2} \frac{\gamma^2}{\sqrt{\beta^2+\gamma^2}\,(\sqrt{\beta^2+\gamma^2}+\gamma)} \qquad (\gamma > 0) \qquad (3.1)$$

Similarly, we obtain the inequalities

$$\sum_{p=1,3,\ldots}^{\infty} \frac{\gamma^3}{(\gamma^2+p^2)(\gamma^2+\beta^2+p^2)} < \frac{\pi}{4} \frac{\gamma^2}{\sqrt{\gamma^2+\beta^2}\,(\sqrt{\gamma^2+\beta^2}+\gamma)} \qquad (\gamma > 0) \qquad (3.2)$$

$$\sum_{p=2,4,\ldots}^{\infty} \frac{\gamma^3}{(\gamma^2+p^2)(\gamma^2+\beta^2+p^2)} < \frac{\pi}{4} \frac{\gamma^2}{\sqrt{\gamma^2+\beta^2}\,(\sqrt{\gamma^2+\beta^2}+\gamma)} \qquad (\gamma > 0) \qquad (3.3)$$

Also, we evaluate from above the sum of the series

$$J_1 = \sum_{p=1}^{\infty} \frac{\gamma^2 p}{(\gamma^2+p^2)^2} \qquad (3.4)$$

Its generating function

$$f(x) = \frac{x}{(\gamma^2+x^2)^2}$$

for $x > 0$ has one maximum extremum at the point $x_1 = \gamma\sqrt{3}$ and one inflection at the point $x_2 = \gamma$. The graph of this function for $0 \le x \le \gamma$ is a convex curve. Within the interval $\gamma \le x < +\infty$ the function monotonically decays. We denote the integral part of γ by $h = [\gamma]$. In evaluating from above the sum of series (3.4) we use the trapezoidal formula on the convex part of the curve $f(x)$ and the rectangle formula on the remainder. We then logically have three possibilities: (1) $x_1 \le h \le \gamma$; (2) $h < x_1 < \gamma$, but $f(h) \ge f(h+1)$; (3) $h < x_1 < \gamma$, but $f(h) < f(h+1)$. It is not difficult to show that in all three cases the following inequality holds:

$$J_1 < \frac{\gamma^2}{2(\gamma^2+1)} + \frac{\gamma^2}{2} f(x_1) + \gamma^2 \int_1^{+\infty} f(x)\,dx$$

Substituting the value of $f(x_1)$ and evaluating the integral, we obtain

$$\sum_{p=1}^{\infty} \frac{\gamma^2 p}{(\gamma^2+p^2)^2} < \frac{1}{2} + \frac{3\sqrt{3}}{32\gamma} - \frac{1}{2(\gamma^2+1)^2} \qquad (\gamma > 0) \qquad (3.5)$$

Similarly, we can derive the inequalities

$$\sum_{p=1,3,\ldots}^{\infty} \frac{\gamma^2 p}{(\gamma^2+p^2)^2} < \frac{1}{4} + \frac{1}{4(\gamma^2+1)} + \frac{3\sqrt{3}}{32\gamma} - \frac{1}{2(\gamma^2+1)^2} \qquad (\gamma > 0) \qquad (3.6)$$

$$\sum_{p=2,4,\ldots}^{\infty} \frac{\gamma^2 p}{(\gamma^2+p^2)^2} < \frac{1}{4} + \frac{3\sqrt{3}}{32\gamma} - \frac{1}{4(\gamma^2+1)^2} \qquad (\gamma > 0) \qquad (3.7)$$

Other estimates of the left-hand sides of (3.5),(3.6) are given in [12].

4. Proceeding to the study of the infinite systems we can estimate from above the sum of the moduli of the coefficients of the infinite systems (2.23), denoting this sum by $T_{mn}^{(1)}$. We find that

$$T_{mn}^{(1)} = \sum_{p=0}^{\infty} H_{mnp}^{(11)} + \sum_{s=0}^{\infty} H_{nsm}^{(12)} \qquad (m, n = 1, 2, \ldots) \tag{4.1}$$

The cases when m or n are zero will be dealt with separately

$$T_{0n}^{(1)} = \sum_{p=0}^{\infty} H_{0np}^{(11)} + \sum_{s=1}^{\infty} H_{ns0}^{(12)}, \quad T_{m0}^{(1)} = \sum_{p=1}^{\infty} H_{m0p}^{(11)} + \sum_{s=0}^{\infty} H_{0sm}^{(12)} \ (m, n = 1, 2 \ldots) \tag{4.2}$$

Substituting (2.27),(2.28),(2.31) into (4.1) and removing terms with zero indices, we obtain

$$T_{mn}^{(1)} = t_{mn}^{(11)} + t_{mn}^{(12)} + r_{mn}^{(12)} \qquad (m, n = 1, 2, \ldots) \tag{4.3}$$

Here

$$t_{mn}^{(11)} = \frac{4}{aL_{mn}^{(1)}} \left\{ \sum_{p=1}^{\infty} \frac{\beta_m^2 \alpha_p^2}{\beta_{np} (\alpha_{mn}^2 + \alpha_p^2)^2} + \sum_{s=1}^{\infty} \frac{\gamma_n^2 \alpha_s^2}{\gamma_{sm} (\alpha_{mn}^2 + \alpha_s^2)^2} \right\}$$

$$t_{mn}^{(12)} = \frac{4\sigma}{aL_{mn}^{(1)}} \left\{ \sum_{p=1}^{\infty} \frac{\gamma_n^2 \beta_{np}}{\beta_{np}^2 (\alpha_{mn}^2 + \alpha_p^2)} + \sum_{s=1}^{\infty} \frac{\beta_m^2 \gamma_{sm}}{\gamma_{sm}^2 (\alpha_{mn}^2 + \alpha_s^2)} \right\}$$

$$r_{mn}^{(12)} = \frac{4\sigma (\gamma_n + \beta_m)}{aL_{mn}^{(1)} \alpha_{mn}^2}$$

Using the inequalities

$$\beta_{np} \geqslant \alpha_p, \quad \gamma_{sm} \geqslant \alpha_s, \quad L_{mn}^{(1)} > \coth \alpha_{mn} a > 1, \quad \gamma_n < \beta_{np} \tag{4.4}$$

and the expressions (2.13) we evaluate $t_{mn}^{(11)}$

$$t_{mn}^{(11)} < \frac{4}{a} \left\{ \sum_{p=1}^{\infty} \frac{\beta_m^2 \alpha_p}{(\alpha_{mn}^2 + \alpha_p^2)^2} + \sum_{s=1}^{\infty} \frac{\gamma_n^2 \alpha_s}{(\alpha_{mn}^2 + \alpha_s^2)^2} \right\} = \frac{4}{\pi} \sum_{p=1}^{\infty} \frac{(\alpha_{mn} a / \pi)^2 p}{[(\alpha_{mn} a / \pi)^2 + p^2]^2}$$

or, from (3.5),

$$t_{mn}^{(11)} < \frac{2}{\pi} + r_{mn}^{(13)}, \qquad r_{mn}^{(13)} = \frac{3\sqrt{3}}{8a\alpha_{mn}} - \frac{2}{\pi [(\alpha_{mn} a / \pi)^2 + 1]^2} \tag{4.5}$$

Also, using the inequalities

$$\frac{\gamma_n \beta_{np}}{\beta_{np}^2} \leqslant \frac{\gamma_n^2 + \beta_{np}^2}{2\beta_{np}^2}, \qquad \frac{\beta_m \gamma_{sm}}{\gamma_{sm}^2} \leqslant \frac{\beta_m^2 + \gamma_{sm}^2}{2\gamma_{sm}^2} \tag{4.6}$$

the expressions (2.13), the inequality (3.1) and the identity

$$\frac{2}{\pi} \sum_{p=1}^{\infty} \frac{x}{x^2 + p^2} = \coth \pi x - \frac{1}{\pi x} \tag{4.7}$$

we can evaluate $t_{mn}^{(12)}$

$$t_{mn}^{(12)} \leqslant \frac{2\sigma}{aL_{mn}^{(1)}} \left\{ \sum_{p=1}^{\infty} \frac{\gamma_n (\gamma_n^2 + \beta_{np}^2)}{\beta_{np}^2 (\alpha_{mn}^2 + \alpha_p^2)} + \sum_{p=1}^{\infty} \frac{\beta_m (\beta_m^2 + \gamma_{pm}^2)}{\gamma_{pm}^2 (\alpha_{mn}^2 + \alpha_p^2)} \right\} =$$

$$= \frac{2\sigma}{\pi L_{mn}^{(1)}} \sum_{p=1}^{\infty} \left\{ \frac{(\gamma_n a / \pi)^3}{[(\gamma_n a / \pi)^2 + p^2] [(\alpha_{mn} a / \pi)^2 + p^2]} + \frac{\gamma_n a / \pi}{(\alpha_{mn} a / \pi)^2 + p^2} + \right.$$

$$+\frac{(\beta_m a/\pi)^3}{[(\beta_m a/\pi)^2+p^2][(\alpha_{mn}a/\pi)^2+p^2]}+\frac{\beta_{mn}a/\pi}{(\alpha_{mn}a/\pi)^2+p^2}\Bigg\}=$$

$$=\frac{\sigma}{L_{mn}^{(1)}}\left\{\frac{1}{\alpha_{mn}}\left[\frac{\gamma_n^2}{\alpha_{mn}+\gamma_n}+\frac{\beta_m^2}{\alpha_{mn}+\beta_m}\right]+\frac{\gamma_n+\beta_m}{\alpha_{mn}}\coth\alpha_{mn}a-\frac{\gamma_n+\beta_m}{a\alpha_{mn}^2}\right\}$$

We can easily establish the inequalities

$$\frac{1}{\sqrt{x^2+y^2}}\left(\frac{x^2}{\sqrt{x^2+y^2}+x}+\frac{y^2}{\sqrt{x^2+y^2}+y}\right)\leqslant\frac{\sqrt{2}}{1+\sqrt{2}}$$

$$\frac{x+y}{\sqrt{x^2+y^2}}\leqslant\sqrt{2}\qquad\begin{pmatrix}x>0\\ y>0\end{pmatrix}$$

Then

$$t_{mn}^{(12)}<\frac{\sigma}{L_{mn}^{(1)}}\left\{\frac{\sqrt{2}}{1+\sqrt{2}}+\sqrt{2}\coth\alpha_{mn}a-\frac{\gamma_n+\beta_m}{a\alpha_{mn}^2}\right\}$$

or, by virtue of (4.4),

$$t_{mn}^{(12)}<2\sigma+r_{mn}^{(14)},\qquad r_{mn}^{(14)}=-\frac{\sigma(\gamma_n+\beta_m)}{L_{mn}^{(1)}a\alpha_{mn}^2}\tag{4.8}$$

From (4.3),(4.5) and (4.8) we obtain the inequality (4.9)

$$T_{mn}^{(1)}<2\sigma+\frac{2}{\pi}+r_{mn}^{(11)}\quad(m,n=1,2,\ldots),\qquad r_{mn}^{(11)}=r_{mn}^{(12)}+r_{mn}^{(13)}+r_{mn}^{(14)}$$

Further, by substituting (2.27),(2.28),(2.31) into (4.2) and using successively inequalities (4.4), expressions (2.13),(4.7) and inequality (3.5), we can evaluate $T_{0n}^{(1)}$

$$T_{0n}^{(1)}=\sum_{p=0}^{\infty}\frac{2\sigma\gamma_n^2}{aL_{0n}^{(1)}\beta_{np}^3}+\frac{4}{aL_{0n}^{(1)}}\sum_{s=1}^{\infty}\frac{\gamma_n^2\alpha_s}{(\gamma_n^2+\alpha_s^2)^2}<\frac{2\sigma}{a\gamma_nL_{0n}^{(1)}}+$$

$$+\frac{2\sigma}{aL_{0n}^{(1)}}\sum_{p=1}^{\infty}\frac{\gamma_n}{\gamma_n^2+\alpha_p^2}+\frac{4}{a}\sum_{s=1}^{\infty}\frac{\gamma_n^2\alpha_s}{(\gamma_n^2+\alpha_s^2)^2}=\frac{2\sigma}{a\gamma_nL_{0n}^{(1)}}+$$

$$+\frac{2\sigma}{\pi L_{0n}^{(1)}}\sum_{p=1}^{\infty}\frac{\gamma_na/\pi}{(\gamma_na/\pi)^2+p^2}+\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{(\gamma_na/\pi)^2s}{[(\gamma_na/\pi)^2+s^2]^2}=$$

$$=\frac{2\sigma}{a\gamma_nL_{0n}^{(1)}}+\frac{\sigma}{L_{0n}^{(1)}}\left[\coth\gamma_na-\frac{1}{\gamma_na}\right]+\frac{4}{\pi}\sum_{s=1}^{\infty}\frac{(\gamma_na/\pi)^2s}{[(\gamma_na/\pi)^2+s^2]^2}<$$

$$<\frac{\sigma}{a\gamma_nL_{0n}^{(1)}}+\sigma+\frac{4}{\pi}\left[\frac{1}{2}+\frac{3\sqrt{3}\pi}{32\gamma_na}\right]$$

$$T_{0n}^{(1)}<\sigma+\frac{2}{\pi}+r_{0n}^{(11)}\quad(n=1,2,\ldots),\qquad r_{0n}^{(11)}=\frac{\sigma}{\gamma_naL_{0n}^{(1)}}+\frac{3\sqrt{3}}{8\gamma_na}\tag{4.10}$$

Similarly we can obtain an estimate for $T_{m0}^{(1)}$:

$$T_{m0}^{(1)}<\sigma+\frac{2}{\pi}+r_{m0}^{(11)}\quad(m=1,2,\ldots),\qquad r_{m0}^{(11)}=\frac{\sigma}{\beta_maL_{m0}^{(1)}}+\frac{3\sqrt{3}}{8\beta_ma}\tag{4.11}$$

For all σ in the range $0<\sigma\leq 0.18$ the inequality

$$2\sigma+\frac{2}{\pi}<0.9968,\qquad\sigma+\frac{2}{\pi}<0.9968$$

holds. Therefore, from inequalities (4.9) to (4.11) we obtain the general estimate

$$T_{mn}^{(1)} < 0.9968 + r_{mn}^{(11)} \qquad (m, n = 0, 1, 2, \ldots;\ 0 < \sigma \leqslant 0.18) \qquad (4.12)$$

where $r_{mn}^{(11)}$ is determined from Formulas (4.9) to (4.11). Obviously

$$r_{mn}^{(11)} = O\left((m^2 + n^2)^{-1/2}\right)$$

We denote the sum of the moduli of the coefficients of the infinite systems (2.24) to (2.26), respectively, by $T_{np}^{(2)}$, $T_{pm}^{(3)}$, $T_{pm}^{(4)}$. Using the inequalities of Section 3, and performing analogous operations, we obtain the following analogous estimates:

$$\begin{aligned} T_{np}^{(2)} &< 0.9968 + r_{np}^{(21)} \qquad (n, p = 0, 1, 2, \ldots) \\ T_{pm}^{(3)} &< 0.9968 + r_{pm}^{(31)} \qquad (p, m = 0, 1, 2, \ldots) \qquad (0 < \sigma \leqslant 0.18) \qquad (4.13) \\ T_{pm}^{(4)} &< 0.9968 + r_{pm}^{(41)} \qquad (p, m = 0, 1, 2, \ldots) \end{aligned}$$

$$r_{np}^{(21)} = O\left((n^2 + p^2)^{-1/2}\right), \qquad r_{pm}^{(31)} = O\left((p^2 + m^2)^{-1/2}\right), \qquad r_{pm}^{(41)} = O\left((p^2 + m^2)^{-1/2}\right)$$

The estimates (4.12) and (4.13) show that for all values of Poisson's ratio σ within the range $0 < \sigma \leq 0.18$ the infinite systems (2.23) to (2.26) are at least quasi-fully regular. The absolute terms (2.29), (2.30) of the infinite systems are bounded. Therefore, if the solution of the corresponding infinite systems is unique, there exists a unique bounded solution to the infinite systems under consideration [13].

Note that to prove the quasi-full regularity of the infinite systems for $0 < \sigma \leq 0.18$ we used the very crude inequalities (4.4) and (4.6). This Allows us to hope that for values of σ which satisfy the inequality $0.18 < \sigma \leq 0.5$ the infinite systems obtained possess the property of regularity.

5. In [5] Teodorescu has considered the particular case of the problem solved above, when the papallelepiped is loaded only by identically distributed normal stresses $p(x, y)$ on two opposite faces $z = \pm l$. The problem was reduced to three infinite systems of linear algebraic equations. We quote one of these, which in the reference cited was numbered (4.19):

$$A_{mn} = -\frac{1}{\lambda_{mn} a \xi_{mn}} \left[\sum_{i=0}^{\infty} h_{imn} B_{ni} + \sum_{i=0}^{\infty} g_{nim} C_{im}\right] \qquad \begin{pmatrix} m, n = 0, 1, \ldots \\ m \neq 0 \text{ for } n = 0 \end{pmatrix} \quad (5.1)$$

$$h_{imn} = \frac{4\alpha_i^2 \beta_m^2}{(\lambda_{mn}^2 + \alpha_i^2)^2} + \frac{4\nu\gamma_n^2}{\lambda_{mn}^2 + \alpha_i^2}, \qquad g_{nim} = \frac{4\gamma_n^2 \alpha_i^2}{(\lambda_{mn}^2 + \alpha_i^2)^2} + \frac{4\nu\beta_m^2}{\lambda_{mn}^2 + \alpha_i^2} \quad (5.2)$$

$$\zeta_{mn} = \coth \lambda_{mn} a + \frac{\lambda_{mn} a}{\sinh^2 \lambda_{mn} a}, \qquad \lambda_{mn}^2 = \beta_m^2 + \gamma_n^2, \qquad \alpha_i = \frac{i\pi}{a} \quad (5.3)$$

where ν is Poisson's ratio. The author asserts that this system is regular and has a unique bounded solution and can be studied on the basis of the same procedures as those outlined by Kaliski in [14]. This assertion is erroneous. In Kaliski's paper completely different infinite systems are obtained and their investigation has no relation to the system (5.1). We shall show that the infinite system (5.1), on the contrary, is not regular. For, if we evaluate the sum of the moduli of its coefficients, we find

$$T_{mn} = \frac{1}{\lambda_{mn} a \xi_{mn}} \sum_{i=0}^{\infty} [h_{imn} + g_{nim}] = \frac{4}{\lambda_{mn} a \xi_{mn}} \sum_{i=0}^{\infty} \left\{\frac{\lambda_{mn}^2 (\pi i/a)^2}{[\lambda_{mn}^2 + (\pi i/a)^2]^2} + \frac{\nu \lambda_{mn}^2}{\lambda_{mn}^2 + (\pi i/a)^2}\right\} =$$

$$= \frac{4\nu}{\lambda_{mn} a \xi_{mn}} + \frac{4}{\pi \xi_{mn}} \sum_{i=1}^{\infty} \left\{\frac{(\lambda_{mn} a/\pi) i^2}{[(\lambda_{mn} a/\pi)^2 + i^2]^2} + \frac{\nu \lambda_{mn} a/\pi}{(\lambda_{mn} a/\pi)^2 + i^2}\right\}$$

Using the identities

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{xn^2}{(x^2+n^2)^2}=\coth \pi x-\frac{\pi x}{\sinh^2 \pi x}, \qquad \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{x}{x^2+n^2}=\coth \pi x-\frac{1}{\pi x}$$

and the notation (5.3), we obtain

$$T_{mn}=\frac{1}{\xi_{mn}}\left\{\coth \lambda_{mn} a-\frac{\lambda_{mn}a}{\sinh^2 \lambda_{mn}a}+2\nu\left[\coth \lambda_{mn}a-\frac{1}{\lambda_{mn}a}\right]\right\}+\frac{4\nu}{\lambda_{mn}a\xi_{mn}}=$$
$$=1+2\nu+\frac{2\nu}{\lambda_{mn}a\xi_{mn}}-\frac{2(1+\nu)\lambda_{mn}a}{\xi_{mn}\sinh^2 \lambda_{mn}a}\quad (m,\ n=1,\ 2,\ \ldots)$$

We see now that the infinite system (5.1) is not regular for all values of Poisson's ratio $0 < \nu \le 0.5$. In exactly the same way we can establish the irregularity of the other two infinite systems. It follows that the assertions contained in [5] concerning the order of the solutions of the infinite systems are without foundation.

BIBLIOGRAPHY

1. Lamé, G., Leçons sur la théorie mathematique de l'élasticité des corps solides. Paris, 1852.
2. Filonenko-Borodich, M.M., Zadacha o ravnovesii uprugogo parallelepipeda pri zadannykh nagruzkakh na ego graniakh (The problem of the equilibrium of an elastic parallelepiped with prescribed loads on its faces). *PMM* Vol.15, № 2, 1951.
3. Filonenko-Borodich, M.M., Dve zadachi o ravnovesii uprugogo parallelepipeda (Two problems on the equilibrium of an elastic parallelepiped). *PMM* Vol.15, № 5, 1951.
4. Mishonov, M., Obshch metod za reshenie na prostranstvenata zadacha na elastichnostta za paralelepipeda. Izv.Tekhn.in-t B"lg.AN, kn.9-10, 15-61, 1960.
5. Teodorescu, P.P., Sur le problème du parallelepipede élastique. Arch. mech.stosowanej, Vol.12, №№ 5-6. 1960.
6. Valov, G.M., Zadacha o szhatii priamougol'nogo parallelepipeda, na osnovaniiakh kotorogo zadany normal'nye peremeshcheniia i kasatel'nye napriazheniia, a na bokovoi poverkhnosti – napriazheniia. (The problem of the compression of a rectangular parallelepiped on the ends of which normal displacements and shear stresses are specified, and on the lateral surface stresses). Nauchn.dokl.vyssh.shkoly, Fiz.-matem.nauki, № 2, 1959.
7. Valov, G.M., Zadacha o ravnovesii priamougol'nogo parallelepipeda, na osnovaniiakh kotorogo zadany normal'nye napriazheniia i kasatel'nye peremeshcheniia, a na bokovoi poverkhnosti – napriazheniia (The problem of the equilibrium of a rectangular parallelepiped on the ends of which normal stresses and tangential displacements are prescribed and on the lateral surface, stresses). Nauchn.dokl.vessh.shkoly. Fiz.-matem.nauki, № 4, 1958.
8. Valov, G.M., Ob odnoi zadache o ravnovesii priamougol'nogo parallelepipeda so smeshannymi granichnymi usloviiami (On a problem of the equilibrium of a rectangular parallelepiped with mixed boundary conditions). Vestn.Mosk.un-ta, № 3, 1959.
9. Baida, E.N., Zadacha ob uprugo-deformirovannom sostoianii ortotropnogo i izotropnogo parallelepipedov (The problem of the elastically deformed state of orthotropic and isotropic parallelepipeds). Izv. vyssh.uchebn.zaved., stroit. i arkh., № 6, 1959.
10. Baida, E.N., Obshchie resheniia teorii uprugosti i zadachi o parallelepipede i tsilindre (General Solutions in the Theory of Elasticity and the Problems of a Parallelepiped and Cylinder). Gosstroiizdat, 1961.
11. Hata Kin-ichi, Some remarks on the three-dimensional problems concerned with the isotropic and anisotropic elastic solids. Mem.Fac.Eng. Hokkaido Univ., Vol.10, № 2, 1956.

12. Abramian, B.L., K ploskoi zadache teorii uprugosti dlia priamougol'nika (On the plane problem in the theory of elasticity for a rectangle). *PMM* Vol.21, № 1, 1957.

13. Kantorovich, L.V. and Krylov, V.I., Priblizhennye metody vysshego analiza (Approximate Methods of Higher Analysis). Gostekhteoretizdat, 1950.

14. Kaliski, S., The dynamical problem of the rectangular parallelepiped. Arch.mech.stosowanej, Vol.10, 1958.

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